For convenience, the results of this paper and its predecessor are here recapitulated.

Gas.	Weak component polarisation (strong component = 100).
Argon	0·46
Hydrogen	3·83
Nitrogen	4·06
Air	5·0
Oxygen	9·4
Carbon dioxide	11·7
Nitrous oxide	15·4
Helium	Not definitely determined, but $< 6·5$

The total intensities of light scattered by helium and by air have been compared. The ratio found was 0·0170. Considering the difficulty of the experiment, this is considered to agree within the limit of error with the ratio of the squares of refractivities, which is 0·0144.

Reduction of Error by Linear Compounding.

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(Abstract.)

If $u_1, u_2, u_3, \ldots$ are observed values of a single quantity U—if, in other words, there are observed values of U containing errors $u_1 - \mathrm{U}$, $u_2 - \mathrm{U}$, $u_3 - \mathrm{U}, \ldots$ —, we may take as an approximate value of U any function of the u's which, if each of the u's in it were replaced by U, would itself become identically U. Usually this function is a linear compound (linear function) of the u's, and is called a weighted mean: and we regard as the best weighted mean that in which the constants of the function are chosen so that it shall have the least possible mean square of error. The principle applies also when the u's, instead of being observations of a single quantity U, are observations of a set of quantities $\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3, \ldots$; with the modification that the effect of replacing (say) u_1 by a function of the u's will be that, if the u's in the function were put equal to the corresponding U's, the function would not become identically equal to U_1, but only approximately equal to

it. In the particular case in which the differences of the u's successively diminish, a permissible form of the function is the sum of u_1 and a linear compound of the differences of the u's of sufficiently high order.

The problem considered in this paper is the slightly more general one of improving a quantity by adding to it a linear compound of certain other quantities, called auxiliaries; the improved value being taken to be the value of this sum when, by suitable choice of the coefficients of the auxiliaries, its m.s.e. (mean square of error) is a minimum. Allied to this, and leading to a similar solution, is the problem of fitting to a set of observations $u_1, u_2, u_3, \ldots$, by the method of least squares, a linear compound of a smaller number of functions of 1, 2, 3, ..., with coefficients to be determined.

The most simple case is that in which the errors of the u's are independent —or at any rate the m.p.e. (mean product of errors) of every pair of different u's is 0—and the m.ss.e. are all equal to 1. The errors are then said to form a standard system. In two papers published a few years ago, on "Reduction of Errors by Means of Negligible Differences" and "Fitting of Polynomial by Method of Least Squares," I investigated the problem for this class of cases, the auxiliaries being taken to be the differences of the u's beyond those of a certain order. The scope of the present paper is wider, and the methods also are simpler. The simplification is effected in two ways: by brief statement of general theorems, and by a theory of conjugate sets of observations.

The fundamental theorem for reduction of error is that the m.p.e. of the improved value and each of the auxiliaries is zero. This leads not only to other general theorems, but also to simplification in dealing with particular problems. Suppose, for instance, that we require the m.s.e. of an improved value, the m.ss.e. and m.pp.e. of the original value and the auxiliaries being known. If we construct the m.s.e. in the usual manner, the terms will form a double series, and will involve all these m.ss.e. and m.pp.e. But the above theorem shows that the m.s.e. of the improved value is equal to the m.p.e. of the improved value and the original value, so that we have only to deal with a single series, involving the m.s.e. of the original value and its m.p.e. with each of the auxiliaries.

In the case of the standard system, mentioned above, the m.s.e. of each u is 1, and the m.p.e. of each pair of u's is 0. For other cases, we construct a somewhat analogous system by taking another set of quantities $y_1, y_2, y_3, \ldots$, equal in number to the u's and connected with them by linear relations, and such that if we place the two sets in linear correspondence, thus:—

$$u_1, u_2, u_3, \ldots,$$
$$y_1, y_2, y_3, \ldots,$$

the m.p.e. of corresponding quantities of the two sets will be 1 and the m.p.e. of quantities which do not correspond will be 0. Sets so related are called conjugate sets. In the case of a standard system the y's are identical with the u's, so that the latter may be called a self-conjugate set.

If there is a second set of quantities $\delta_1, \delta_2, \delta_3, \ldots$, also connected with the u's by linear relations, there are converse linear relations between the conjugates of the δ's and of the u's. In particular, if the δ's are successive differences of the u's, and if the conjugates of the δ's are $\sigma_1, \sigma_2, \sigma_3, \ldots$, the σ's are sums, of successive orders, of the y's. The improved values of any specified δ's, using the remaining δ's as auxiliaries, are linear compounds of the σ's which correspond to the former. Hence, if the u's are a self-conjugate set, and the δ's are their differences, the improved values of any linear compounds of the u's, using certain δ's as auxiliaries, are linear compounds of the σ's which correspond to the other δ's. Since the moments of the u's, like the σ's, are linear compounds of successive sums of the u's, this means that the improved values are linear compounds of the moments; and it explains the appearance of the moments in this connection.

The earlier part of the paper deals with the main theorems relating to conjugate sets and to reduction of error, and with the relation of reduction of error to fitting. Certain general formulæ are then established, by means of which the solution of the problem for any particular type of data can be completed if the values of certain quantities can be found. The paper concludes with the application of the general formulæ to the particular case of the self-conjugate set. Some of the formulæ thus obtained are, of course, equivalent to formulæ obtained in the two earlier papers; but the methods of these are to a large extent superseded by the more general and more compact methods of the present paper.